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FREE FERMION SOLUTION FOR DIMER PROBLEM

R.Hayn*, V.N.Plechko

The new noncombinatorial approach to the dimer problem based on integration over fermionic fields is presented. The partition function of the closed-packed dimer model on the inhomogeneous rectangular lattice is obtained in the form of the Gaussian integral over Grassmann variables.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR and Max Planck Institut für Festkörperforschung, Stuttgart, Germany.

Решение задачи димеров и свободные фермионы

Р.Хайн, В.Н.Плечко

Предложен новый некомбинаторный подход к задаче димеров, основанный на интегрировании по фермионным полям. Для статистической суммы плотноупакованной димерной модели на неоднородной прямоугольной решетке найдено выражение в виде гауссовского интеграла по грассмановым переменным.

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1. Introduction

The closed-packed dimer model on the homogeneous rectangular lattice was first solved by Kasteleyn [1] and Temperley and Fisher [2]. Besides its importance for the understanding of the dimer combinatorics itself, this remarkable solution has contributed much to the theory of the 2D Ising model and the other lattice problems in statistical mechanics [3—5]. The dimer models have also found physical applications in different branches. We mention here the connection of the dimer problem on the hexagonal lattice [6] to commensurate-incommensurate phase transitions due to a domain-wall analogy [7,8]. This also provides us with a model for the Pokrovsky — Talapov phase transition [9]. The second line of interest arises from the RVB theory of high-T_c superconductivity [10—12]. The classical dimer model is here a structural ingredient of the theory to describe the short range resonating valence bond state [11,12]. The important feature of the dimer model is its fermionic nature. Kasteleyn [1,6] used combinatorial argu-

*Max Planck Institut für Festkörperforschung, 7000 Stuttgart 80, Germany

ments to show that the statistical sum of dimer problems can be expressed in terms of Pfaffian forms. In retrospection, this already suggests the fermionic nature of the problem. The combinatorial fermionic analysis of the problem has been performed later on by Samuel [13] and Abanov [14]. However, the combinatorial approaches are rather complicated and differ significantly from the methods commonly accepted in solid state physics where the dimer models have found recently important applications.

In this note we present a new constructive solution for the 2D dimer problem based on the Grassmann variable (fermionic) integrals and factorization principles for the density matrix. The approach is straightforward and very simple, combinatorial considerations are not needed. We present the basic ideas of fermionization by an example of the rectangular inhomogeneous lattice. For this lattice, we derive a representation for the partition function in the form of a Gaussian fermionic integral. This means that the dimer model is equivalent to the free fermion field theory on a lattice. The principal point of the solution is the mirror-ordered representation for the dimer density matrix. At this stage we apply the ideas first developed in the context of the 2D Ising model [15].

2. The Closed-Packed Dimer Problem

We consider the closed-packed dimer problem on the inhomogeneous rectangular lattice with free boundary. The lattice sites mn are numbered by discrete Cartesian coordinates $m = 1, 2, ..., M$ and $n = 1, 2, ..., N$ running in horizontal and vertical directions, respectively. The dimers are objects living on lattice bonds. The given bond may be either free or covered by a dimer together with the two adjacent lattice sites. A closed-packed dimer covering is such that each lattice site is occupied by exactly one dimer. The lattice must have even number of sites to be covered completely in a closed-packed fashion since each dimer covers exactly two sites. An example of a closed-packed dimer configuration for a rectangular lattice is shown in Fig.1a. We will associate the horizontal bond $(mn|m+1n)$ and the vertical bond $(mn|mn+1)$ with the site mn, as it is shown in Fig.1c. Let $t^{(1)}_{mn}$ be the weight of the horizontal mn-bond covered by a dimer, analogously we define the weight $t^{(2)}_{mn}$ for the vertical mn-bond, see Fig.1c. The weight of a free bond is 1. The Boltzmann weight of a configuration is the product of the activities $t^{(\alpha)}_{mn}$ $(\alpha = 1,2)$ of all occupied bonds and the partition function Q is the sum over all possible closed-packed configurations. This partition function (or generating function in combinatorial interpretation) is the main subject of interest.

The formal combinatorial definition for Q can be put in a more constructive form as follows (also see [16]). With each lattice site mn we associate the commuting nilpotent variable η_{mn} with the property $\eta_{mn}^2 = 0$ and write:

$$Q = \underset{(\eta)}{Sp} \prod_{m=1}^{M} \prod_{n=1}^{N} (1 + t_{mn}^{(1)} \eta_{mn} \eta_{m+1\,n})\,(1 + t_{mn}^{(2)} \eta_{mn} \eta_{m\,n+1}), \qquad (1)$$

with the free-boundary conditions: $\eta_{M+1\,n} = \eta_{m\,N+1} = 0$. The averaging over separable η_{mn} variable is here defined as follows:

$$\underset{(\eta_{mn})}{Sp}\,(1|\eta_{mn}|\eta_{mn}^2|\eta_{mn}^3|\eta_{mn}^4|...) = (0|1|0|0|0|...), \qquad (2)$$

and the total averaging in (1) involves the averagings over η_{mn} at all sites. By multiplication of the factors in the «density matrix» in (1) we get a polynomial in η-variables in which the given η_{mn} variable enters up to the fourth power. To fulfil the closed-packing condition we have to pick up only the terms in which each of the variables η_{mn} is presented in the first power. This is just realized by the selection rules (2). Our goal is now to pass from the commuting variables η to the anticommuting Grassmann variables a with the corresponding change of averaging. The point is that the Grassmann variables are «good» variables, with many plausible properties, and we know how to extract numbers from the Grassmann-variable expressions, which is not the case for the commuting nilpotent η-variables.

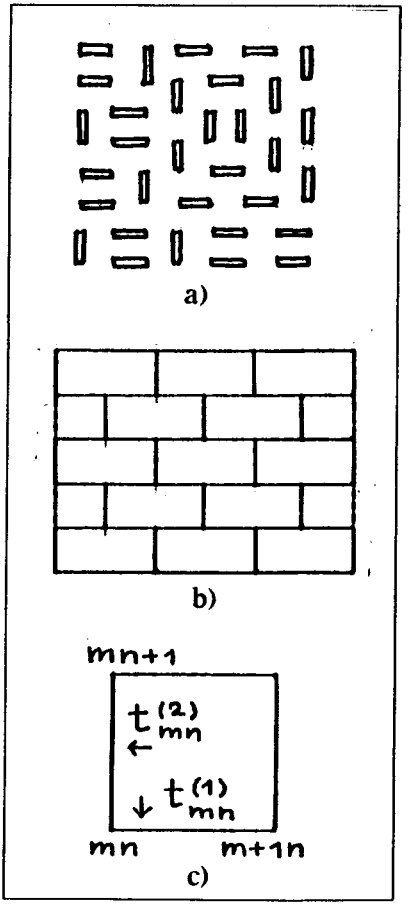

Fig. 1. a) The closed-packed dimer covering for a rectangular lattice. b) A fragment of brick lattice, such a lattice is topologically equivalent to hexagonal lattice. c) Enumeration of sites and bonds on a rectangular cell

3. Grassmann Variables

We remember that the Grassmann variables are classic fermionic numbers purely anticommuting to zero. Given a set of Grassmann variables $a_1,..., a_N$, we have $a_i a_j + a_j a_i = 0$, $a_j^2 = 0$. The Berezin's rules of integration over one variable are [17]:

$$\int da_j \cdot a_j = 1, \quad \int da_j \cdot 1 = 0. \tag{3}$$

In the multidimensional integral the differential symbols $da_1,..., da_N$ are again anticommuting with each other and with the variables. The basic formulas of the Grassmann-variable analysis are for the Gaussian fermionic integrals [17]. The Gaussian integral of the first kind is related to the determinant:

$$\int \prod_{j=1}^{N} da_j^* \, da_j \, e^{\sum_{i=1}^{N} \sum_{j=1}^{N} a_i A_{ij} a_j^*} = \det \hat{A}, \tag{4}$$

here $\{a_j, a_j^*\}$ is a set of completely anticommuting Grassmann variables, the matrix in the exponential is arbitrary. The Gaussian integral of the second kind is related to the Pfaffian form of the skew-symmetric matrix:

$$\int da_N ... da_2 da_1 \, e^{\frac{1}{2}\sum_{i=1}^{N} \sum_{j=1}^{N} a_i A_{ij} a_j} = \text{Pfaff}\, \hat{A}, \quad A_{ij} = -A_{ji}. \tag{5}$$

The Pfaffian form is some combinatorial polynomial in elements A_{ij} known in mathematics for a long time [18]. The Pfaffian and determinant of the associated skew-symmetric matrix are algebraically related: $\det \hat{A} = (\text{Pfaff}\, \hat{A})^2$. This relation can be most easily proved in terms of the fermionic integrals like (4) and (5) [13].

Let it be given two Grassmann variables a and a^*. The elementary Gaussian exponential is $e^{\lambda a a^*} = 1 + \lambda a a^*$ (the series terminates since $(aa^*)^2 = 0$). Making use of the basic rules (3), for the complete set of the Gaussian averages we find: $\int da^* da \, e^{\lambda a a^*} \{1, a, a^*, aa^*\} = \{\lambda, 0, 0, 1\}$. Note that the linear terms give zero result as a consequence of $\int da_j \cdot 1 = 0$. We can then factorize the typical dimer weight in (1) as follows:

$$1 + t\eta\eta' = \int da^* da \, e^{aa^*} (1 + ta\eta)(1 + a^*\eta'). \tag{6}$$

We will apply the factorization like (6) in order to pass in (1) from the commuting η-variables to the anticommuting Grassmann variables.

4. Fermionization

We first factorize the local weights from (1) following (6). To this end, we introduce a set of purely anticommuting Grassmann variables $\{a_{mn}, a^*_{mn}, b_{mn}, b^*_{mn}\}$, a pair per bond, are write:

$$1 + t^{(1)}_{mn}\eta_{mn}\eta_{m+1\,n} = \int da^*_{mn} da_{mn}\, e^{a_{mn}a^*_{mn}} \times$$

$$\times (1 + t^{(1)}_{mn}a_{mn}\eta_{mn})(1 + a^*_{mn}\eta_{m+1\,n}) = \underset{(a_{mn})}{Sp}\, A_{mn}A^*_{m+1\,n}, \qquad (7a)$$

$$1 + t^{(2)}_{mn}\eta_{mn}\eta_{m\,n+1} = \int db^*_{mn} db_{mn}\, e^{b_{mn}b^*_{mn}} \times$$

$$\times (1 + t^{(2)}_{mn}b_{mn}\eta_{mn})(1 + b^*_{mn}\eta_{m\,n+1}) = \underset{(b_{mn})}{Sp}\, B_{mn}B^*_{m\,n+1}, \qquad (7b)$$

where we introduced the abbreviated notation for the arising Grassmann factors:

$$A_{mn} = 1 + t^{(1)}_{mn}a_{mn}\eta_{mn}, \quad A^*_{m+1\,n} = 1 + a^*_{mn}\eta_{m+1\,n},$$

$$B_{mn} = 1 + t^{(2)}_{mn}b_{mn}\eta_{mn}, \quad B^*_{m\,n+1} = 1 + b^*_{mn}\eta_{m\,n+1}, \qquad (8)$$

and by Sp we denote Gaussian averagings like $\int da^* da\, e^{aa^*}(...)$ and $\int db^* db\, e^{bb^*}(...)$ as is clear from above. Note that the indices mn of the Grassmann factors in (7)—(8) are chosen to be equal to the mn indices of the η-variables involved in the corresponding factors.

There are in general the four factors $A_{mn}, A^*_{mn}, B_{mn}, B^*_{mn}$ with a given index mn, which all involve the same variable η_{mn} associated with the mn-site of the lattice. The idea is to group together the four factors with the same η_{mn}-variable and to perform the η-averaging in each group of factors independently. The obstacle is that separable factors (8) are neither commuting, nor anticommuting with each other. Thus we have to make a special consideration of ordering in the products of noncommuting factors.

To solve this problem, we will use a special arrangement of the Grassmann factors. The two main principles are illustrated here by tutorial examples:

$$(x_0\bar{x}_1)\,(x_1\bar{x}_2)\,(x_2\bar{x}_3)\,(x_3\bar{x}_4) = x_0(\bar{x}_1x_1)\,(\bar{x}_2x_2)\,(\bar{x}_3x_3)\,\bar{x}_4, \tag{9}$$

and

$$(x_1\bar{x}_1)\,(x_2\bar{x}_2)\,(x_3\bar{x}_3) = (x_1(x_2(x_3\bar{x}_3)\,\bar{x}_2)\,\bar{x}_1) = x_1x_2x_3\cdot\bar{x}_3\bar{x}_2\bar{x}_1. \tag{10}$$

In (9) we simply reread the product by joining the factors with the same index. In (10) we assume that the doublets $x_j\bar{x}_j$ are totally commuting with any separable factors though the factors themselves may be noncommuting with each other.

We must prepare such an expansion of the dimer density matrix into the product of the noncommuting factors (8) that it is possible to perform the η_{mn}-averaging at each site. To this end, the Grassmann factors with given mn are to be placed nearby in the process of averaging over η_{mn}. What can be used in the ordering arrangements is that the doubled factors $A_{mn}A^*_{m+1\,n}$ and $B_{mn}B^*_{m\,n+1}$ presenting the bond weights in (7) can be considered as totally commuting objects, if taken as a whole, under the sign of the Gaussian averaging. We thus may move symbols $A_{mn}A^*_{m+1\,n}$ and $B_{mn}B^*_{m\,n+1}$ through any product of other Grassmann factors. With these notes we now directly proceed to the construction of a suitable factorized representation for the density matrix. In the forthcoming transformations from (11) to (13) we omit for brevity the symbol of the Gaussian averaging on the right-hand sides.

First, putting one weight between the two Grassmann factors of another, we write:

$$(1 + t^{(1)}_{mn}\eta_{mn}\eta_{m+1\,n})\,(1 + t^{(2)}_{mn}\eta_{mn}\eta_{m\,n+1}) =$$

$$= B_{mn}A_{mn}A^*_{m+1\,n}B^*_{m\,n+1}. \tag{11}$$

Next, we fix n and multiply the factors over m applying the rules (9) and (10):

$$\prod_{m=1}^{M}(1 + t^{(1)}_{mn}\eta_{mn}\eta_{m+1\,n})\,(1 + t^{(2)}_{mn}\eta_{mn}\eta_{m\,n+1}) =$$

$$= \prod_{m=1}^{M}\left\{B_{mn}A_{mn}A^*_{m+1\,n}\middle|B^*_{m\,n+1}\right\} =$$

$$= \prod_{m=1}^{M} \overrightarrow{B_{mn} A_{mn} A^*_{m+1\,n}} \cdot \prod_{m=1}^{M} \overleftarrow{B^*_{m\,n+1}} =$$

$$= \prod_{m=1}^{M} \overrightarrow{A^*_{mn} B_{mn} A_{mn}} \cdot \prod_{m=1}^{M} \overleftarrow{B^*_{m\,n+1}}, \qquad (12)$$

where the arrows indicate the directions of increasing m in the ordered products. In the last line we have taken into account that $A^*_{M+1\,n} = 1$ due to the boundary condition $\eta_{M+1\,n} = 0$, also see the definition of factors in (8). And, on the contrary, on the left end of the ordered product in (12) we introduced, formally, the lacking factors A^*_{1n} in which we put $a^*_{0n} = 0$, so that in fact $A^*_{1n} = 1$.

In turn, we now multiply the products (12) with respect to index n, with n increasing from left to right, and applying (9):

$$\prod_{n=1}^{N} \left\{ \prod_{m=1}^{M} \overrightarrow{A^*_{mn} B_{mn} A_{mn}} \cdot \prod_{m=1}^{M} \overleftarrow{B^*_{m\,n+1}} \right\} =$$

$$= \prod_{n=1}^{N} \left\{ \prod_{m=1}^{M} \overleftarrow{B^*_{mn}} \cdot \prod_{m=1}^{M} \overrightarrow{A^*_{mn} B_{mn} A_{mn}} \right\}, \qquad (13)$$

where in the final expression we again annihilate factors $B^*_{m\,N+1} = 1$ and create factors $B^*_{m1} = 1$ with $b^*_{m0} = 0$, analogously to the boundary transformations in (12). In fact, being forced in (12) to separate the mn and $m\,n+1$ Grassmann factors in order to apply the linear arrangement (9) with respect to m, in (13) we are trying to restore the normal state by gathering the factors with equal values of mn into separable groups.

For the dimer partition function (1) we thus come to the following factorized representation:

$$Q = \underset{(\eta\,|\,a,b)}{Sp} \prod_{n=1}^{N} \left\{ \prod_{m=1}^{M} \overleftarrow{B^*_{mn}} \cdot \prod_{m=1}^{M} \overrightarrow{A^*_{mn} B_{mn} A_{mn}} \right\}, \qquad (14)$$

where we have restored the sign of the total Gaussian averaging over the fermionic variables. Note that the original free-boundary conditions for the η-variables have now been transformed into the free-boundary fermionic conditions in (14). The η-averaging can be readily performed in representation (14).

The η-averaging reduces to the averaging over separable variables η_{mn} at the junction of the m-products in (14) and yields, finally, the product of the linear forms:

$$L_{mn} = t^{(1)}_{mn} a_{mn} + t^{(2)}_{mn} b_{mn} + a^*_{m-1\,n} + (-1)^{m+1} b^*_{m\,n-1}. \qquad (15)$$

We first fix n and perform the η-averaging at the junction for $m = 1,2,..,M$, and all over again for other values of n. At the first step, $m = 1$, we average the product $B^*_{mn}A^*_{mn}B_{mn}A_{mn} = 1 + \eta_{mn}(t^{(1)}_{mn}a_{mn} + t^{(2)}_{mn}b_{mn} + a^*_{m-1\,n} + b^*_{m\,n-1}) + ...$ with the result (15). Then we move the fermionic form L_{mn} from the junction to the left through the remaining product of B^*_{mn} factors, this explains the appearance of the $(-1)^{m+1}$ factor in (15) in the general case. The η-variables being eliminated, the partition function is the product of the forms L_{mn} under the Gaussian fermionic averaging coming from (14). To bring this expression into a more convenient form, we exponentiate the factors L_{mn} introducing auxiliary Grassmann variables c_{mn} and making use of the identity $L_{mn} = \int dc_{mn}\, e^{c_{mn}L_{mn}}$. This gives:

$$\cdot Q = \int \prod_{n=1}^{N} \prod_{m=1}^{M} da^*_{mn} da_{mn} db^*_{mn} db_{mn} \overrightarrow{dc_{mn}} \times$$

$$\times \exp\left\{ \sum_{m=1}^{M} \sum_{n=1}^{N} [a_{mn}a^*_{mn} + b_{mn}b^*_{mn} + c_{mn}L_{mn}] \right\}, \qquad (16)$$

with $a^*_{0n} = 0$, $b^*_{m0} = 0$. We have in (16) the fermionic Gaussian integral. This expression can in turn be simplified by integrating out the a- and b-fields by using the identity $\int da^* da\, e^{aa^* + aL' + a^*L''} = e^{L''L'}$, which follows from the basic rules (3), here L', L'' are some linear forms in c-variables. The remaining integral is:

$$Q = \int \prod_{n=1}^{N} \prod_{m=1}^{M} \overrightarrow{dc_{mn}} \times$$

$$\times \exp\left\{ \sum_{m=1}^{M} \sum_{n=1}^{N} \left[t^{(1)}_{mn} c_{m+1\,n} c_{mn} + (-1)^{m+1} t^{(2)}_{mn} c_{m\,n+1} c_{mn} \right] \right\}, \qquad (17)$$

with free-boundary conditions for the c-variables, $c_{M+1\,n} = 0$, $c_{m\,N+1} = 0$. This is the final exact representation for the partition function of the closed-packed inhomogeneous dimer model disposed on a finite rectangular lattice with free boundary. This expression is completely equivalent to the original representation of eq. (1). The partition function is now expressed as a simple fermionic Gaussian integral. In the field theoretical language the quadratic fermionic form in the exponential is called fermionic action. Since the action is quadratic, we deal here with the free fermion theory for the dimer model.

5. Discussion

The Gaussian representation (17) is our main result. As it follows from (5), the integral (17) defines a Pfaffian form, this fact is in accordance with the Kasteleyn analysis of dimer combinatorics [1]. We stress however that we have simply calculated (17) in a formal way, without using any combinatorial considerations.

It is important that the fermionic representation (17) is obtained in the most general case of an inhomogeneous distribution of the dimer weights. This can be used at least for three purposes: (i) one can express the dimer-dimer correlation functions simply by differentiating (17) with respect to the local weights $t^{(\alpha)}_{mn}$, (ii) one can study the disordered problem or the influence of impurities, and (iii) one can consider in a simple way problems with a complicated structure of the elementary cell. We also note that the representation (17) contains all the information about the brick lattice, see Fig.1b, which is equivalent to the hexagonal lattice neglecting the boundary effects. The fermionic representation for Q for the brick-hexagonal lattice arises simply by making zero some of the vertical weights $t^{(2)}_{mn}$ in accordance with Fig.1b. It is interesting that the homogeneous hexagonal lattice exhibits an exotic phase transition [6—8] as distinct from the homogeneous rectangular lattice with no phase transition. Eq. (17) thus preserves the information about partition functions and correlations in homogeneous rectangular and hexagonal lattices and may be a good starting point also for further studies.

For actual calculations for homogeneous lattices, the standard device is to pass to the momentum space (Fourier transformation for fermions). Let us illustrate this for the rectangular lattice. In the homogeneous case, $t^{(1)}_{mn} = t_1$ and $t^{(2)}_{mn} = t_2$, the fermionic action in (17) can be put to block-diagonal form by the substitution:

$$c_{mn} = \frac{2i^{m+n}}{\sqrt{(M+1)(N+1)}} \sum_{p=1}^{M} \sum_{q=1}^{N} c_{pq} \sin \frac{\pi p m}{M+1} \sin \frac{\pi q n}{N+1}. \qquad (18)$$

Let both M and N be even. The partition function then appears in the form (momentum space representation):

$$Q = \int \prod_{q=1}^{N} \prod_{p=1}^{M} \overrightarrow{dc_{pq}} \exp \sum_{p=1}^{\frac{1}{2}M} \sum_{q=1}^{\frac{1}{2}N} \left\{ 2it_1 \cos \frac{\pi p}{M+1} \times \right.$$

$$\times (c_{pq} c_{M+1-p\,N+1-q} + c_{p\,N+1-q} c_{M+1-pq}) + 2it_2 \cos \frac{\pi q}{N+1} \times$$

$$\left. \times (c_{pq} c_{p\,N+1-q} + c_{M+1-pq} c_{M+1-p\,N+1-q}) \right\}. \qquad (19)$$

The pq sum in (19) is reduced to a half-interval $\frac{1}{2}M, \frac{1}{2}N$ when we select explicitly and join together all the terms including the variables c_{pq}, $c_{M+1-p\,q}$, $c_{p\,N+1-q}$, $c_{M+1-p\,N+1-q}$. We thus have to integrate separately over the groups of variables c_{pq}, $c_{M+1-p\,q}$, $c_{p\,N+1-q}$, $c_{M+1-p\,N+1-q}$. Evaluation of the integral gives:

$$Q = \prod_{p=1}^{\frac{1}{2}M} \prod_{q=1}^{\frac{1}{2}N} \left[4t_1^2 \cos^2 \frac{\pi p}{M+1} + 4t_2^2 \cos^2 \frac{\pi q}{N+1} \right], \qquad (20)$$

in accordance with the combinatorial result [1,2].

The fermionization procedure can also be performed, as a generalization of the treatment exposed above for the free boundary, for the toroidal (periodic in both directions) boundary conditions for the inhomogeneous rectangular lattices. We refer here to the experience with the 2D Ising model [19]. Not going into the detail, the final result is:

$$Q|_{\text{torus}} = \frac{1}{2} [G|_{--} + G|_{-+} + G|_{+-} - G|_{++}], \qquad (21)$$

where the fermionic integral G is the integral (17), but now we have the four different combinations of the aperiodic-periodic closing conditions for fermions: $(\pm|\pm) = (c_{M+1\,n} = \pm c_{1n} | c_{m\,N+1} = \pm c_{m1})$. In the homogeneous case $t_{mn}^{(\alpha)} = t_\alpha$, the four integrals in (21) can again be evaluated explicitly, reproducing in a simple way the result of complicated combinatorial analysis [1].

Note that putting $t_1 = t_2 = 1$ in (20), or in the corresponding analytic expression given by (21) in the homogeneous case, we simply get the number of the dimer configurations on the corresponding lattices. Fisher [20] has evaluated, as illustration, the number of all possible coverings of the checkerboard by dominos, which appears to be $N_F = 12988816$ (this number is given by (20) with $M = N = 8$ and $t_1 = t_2 = 1$). We have evaluated the analogous number for the checkerboard swept into a torus. The torus number appears to be significantly larger, $N_T = 311853312$.

In conclusion, we have reformulated the closed-packed dimer problem on the inhomogeneous 2D lattice as a free fermion field theory. The partition function is obtained in the form of a Gaussiann Grassmann-variable integral. This puts the dimer problem, which is originally a combinatorial problem, more close to the typical models of quantum statistics and solid state physics with the opportunity to apply the well developed field-theoretical methods. In the homogeneous cases fermionization yields exact analytic solutions for thermodynamic functions and correlations. The fermionization procedure can as well be applied to more complicated dimer like problems equivalent to non-Gaussian fermionic theories. Grassmann variables are the powerful tool for studying dimer-type models.

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